

# Supersymmetric Asymptotic AdS and Lifshitz Solutions in Einstein-Weyl and Conformal Supergravities

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## ABSTRACT

We show that there exist supersymmetric Lifshitz vacua in off-shell Einstein-Weyl supergravity, in addition to the BPS AdS<sub>4</sub> vacuum. The Lifshitz exponents are determined by the product of the cosmological constant and the coupling of the Weyl-squared term. We then obtain the equations of the supersymmetric solutions that are asymptotic to the AdS or Lifshitz vacua. We obtain many examples of exact solutions as well as numerical ones. We find examples of extremal AdS black branes whose near-horizon geometry is AdS<sub>2</sub> × T<sup>2</sup>. We also find an extremal Lifshitz black hole with  $z = -2$ , whose horizon coincides with the curvature singularity. However the asymptotic Lifshitz solutions are in general smooth wormholes. In conformal supergravity, we find intriguing examples of non-extremal “BPS” AdS and Lifshitz black holes whose local Killing spinor is divergent on the horizon. We show that all the supersymmetric asymptotic AdS and Lifshitz solutions have the vanishing Noether charge associated with some scaling symmetry. We also study the integrability condition of the Killing spinor equation and the supersymmetric invariance of the action. Finally we show that the only spherically-symmetric BPS solution is the AdS vacuum.

# 1 Introduction

One of the most important advances in string theory is the AdS/CFT correspondence [1, 2, 3], which is a duality between a conformal field theory and a closed string theory on the AdS background. The principle of holography underlying the AdS/CFT has been shown to be applicable for broader classes of gauge/gravity duality outside the string theory. For asymptotic AdS backgrounds, the corresponding field theories are Lorentz invariant. In the context of condensed matter physics, various systems exhibit a dynamical scaling near fixed points

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad z \neq 1. \quad (1)$$

In other words, rather than obeying the conformal scale invariance  $t \rightarrow \lambda t$ ,  $x_i \rightarrow \lambda x_i$ , the temporal and the spatial coordinates scale anisotropically.

Requiring also time and space translation invariance, spatial rotational symmetry, spatial parity and time reversal invariance, the authors of [4] were led to consider  $D$  dimensional geometries of the form

$$ds^2 = \ell^2 \left( -r^{2z} dt^2 + r^2 dx^i dx^i + \frac{dr^2}{r^2} \right). \quad (2)$$

(See also [5].) This metric obeys the scaling relation (1) if one also scales  $r \rightarrow \lambda^{-1} r$ . If  $z = 1$ , the metric reduces to the usual AdS metric in Poincaré coordinates with AdS radius  $\ell$ . Metrics of the form (2) can be obtained as solutions in general relativity with a negative cosmological constant if appropriate matter is included. For example, solutions were found by introducing 1-form and 2-form gauge fields [4]; a massive vector field [6]; in an abelian Higgs model [7]; and with a charged perfect fluid [8]. A class of Lifshitz black hole solutions with non-planar horizons was found in [9, 10]. The string and supergravity embedding has been found in [11]–[18].

Whilst supersymmetric AdS vacua and asymptotic AdS solutions are plenty and can be simple in string and supergravities, few examples of supersymmetric Lifshitz and asymptotic Lifshitz solutions have been constructed and they all involve some complicated hypermultiplets in addition to the much simpler vector multiplet. Although the advantage of supersymmetry in condensed matter system is less evident, the supersymmetry helps to stabilize classical solutions and enables us to check the AdS/CFT correspondence to the higher order. However, the complexity of these BPS Lifshitz solutions makes the task difficult.

Recently, it was shown that Lifshitz vacua and Lifshitz black holes arise naturally in higher-derivative pure gravities [19]. (See also [20, 21, 22] for recent construction of exact

solutions in extended gravities with quadratic curvature terms.) This is related to the fact that higher-order gravities contain massive spin-2 modes as well as the massless graviton. The massive spin-2 mode plays an analogous role of massive vector in supporting the Lifshitz black hole.

Extended gravities with quadratic curvature terms can be supersymmetrized in the off-shell formalism using a chiral superfield [23]. In particular, the Einstein-Weyl supergravity was studied in detail recently [24]. The theory admits a supersymmetric AdS vacuum, and its linear spectrum was obtained. In this paper, we demonstrate that in addition to the AdS vacuum, the Einstein-Weyl supergravity admits BPS Lifshitz solutions that preserve  $\frac{1}{4}$  of the supersymmetry. The Lifshitz exponent  $z$  is determined by the product of the cosmological constant and the coupling of the Weyl-squared term.

The paper is organized as follows. In section 2, we review Einstein-Weyl supergravity. The bosonic field content of the theory consists of the metric as well as a massive vector and a complex scalar. For special choice of parameters, the theory becomes conformal supergravity. In section 3, we obtain Lifshitz solutions. There are two types of such solutions. The first of which were obtained previously in Einstein-Weyl pure gravity [19] and the solutions are not supersymmetric. The second type of solutions are new with the massive vector field turned on. We verify by an explicit construction of Killing spinors that these solutions preserve  $\frac{1}{4}$  of the supersymmetry. In section 4, we obtain the equations governing the supersymmetric backgrounds that are asymptotic to the AdS or the Lifshitz vacua. We obtain many examples of exact solutions in Einstein-Weyl supergravity and conformal supergravity. We then use numerical analysis to discuss the general feature of these solutions. All the asymptotic Lifshitz solutions are plane-symmetric. Such a solution has a new global scaling symmetry and the associated Noether charge is a conserved quantity. We find that owing to the supersymmetry, the BPS solutions we have obtained all have the vanishing Noether charge. We present this analysis in section 5.

An important feature of off-shell supergravity is that one can add additional super invariants to the action without modifying the supersymmetric transformation rules. This makes the supersymmetric invariance of the action of off-shell supergravity very different from that of usual on-shell supergravity. In section 6, we study the integrability condition of the Killing spinor equation, and also the invariance of the action. Some new features arise in off-shell supergravity. In section 7, we consider spherically-symmetric solutions. We demonstrate that only supersymmetric such a solution in Einstein-Weyl supergravity is the AdS vacuum. We conclude the paper in section 8.

## 2 Einstein-Weyl supergravity

The field content of the off-shell  $\mathcal{N} = 1$ ,  $D = 4$  supergravity consists of the metric  $e_\mu^a$ , a massive vector  $A_{(1)}$  and a complex scalar  $S + iP$ , totalling 12 off-shell degrees of freedom, matching with that of the gravitino  $\psi_\mu$ . The general formalism for constructing a supersymmetric action for any chiral superfield was obtained in [23]. For appropriate choices of superfields, one obtains the actions of the supersymmetrizations of the cosmological term, the Einstein Hilbert term and higher-order curvature terms. Excluding the higher-order curvature terms, the massive vector and scalars are both auxiliary and are set to zero by the equations of motion, leading to  $\mathcal{N} = 1$  supergravity in four dimensions. However, when higher-order curvature invariants are included, these auxiliary fields may develop dynamics and cannot be truncated out. At the quadratic order of the curvature, there are two super invariants and one of them is the Weyl-squared invariant. In this paper we shall consider including only this quadratic invariant, which has a consequence that the vector field becomes dynamical whilst the complex scalar remains auxiliary. (The other quadratic invariant will turn on the complex scalar as well.) Adopting the notation of [24], the bosonic Lagrangian is given by

$$e^{-1}\mathcal{L} = R - \frac{2}{3}(A_{(1)}^2 + S^2 + P^2) + 4S\sqrt{-\Lambda/3} + \frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{3}\alpha F_{(2)}^2, \quad (3)$$

where  $C$  is the Weyl tensor and  $F_{(2)} = dA_{(1)}$ . Note that the quantity  $m = \sqrt{-\Lambda/3}$  can take both positive and negative values. The supersymmetric transformation rule for the gravitino is given by

$$\delta\psi_\mu = -D_\mu\epsilon + \frac{1}{6}(2A_\mu - \Gamma_{\mu\nu}A^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5P)\epsilon. \quad (4)$$

The full supersymmetric action and transformation rules will be discussed in section 6. The leading-order supergravity with  $\alpha = 0$  (and also  $\Lambda = 0$ ) was constructed much earlier in [26, 27]. Note that compared to [24], we have sent vector field  $A_{(1)}$  to  $iA_{(1)}$  in (3) and (4), which can be clearly done for auxiliary fields in off-shell supergravities. This is because the vector field, being part of the auxiliary multiplet, is not associated with any central charge in the super algebra, and hence sending  $A_{(1)}$  to  $iA_{(1)}$  does not upset the positiveness of the anti-commutator of the supercharges.

The equations of motion for the scalar fields  $S$  and  $P$  imply that

$$S = 3\sqrt{-\Lambda/3}, \quad P = 0, \quad (5)$$

and hence they are auxiliary with no dynamical degree of freedom. The equations of motion for the vector field and the metric are given by

$$\begin{aligned} 0 &= \alpha \nabla^\mu F_{\mu\nu} + A_\nu, \\ 0 &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{2}{3} \alpha (F_{\mu\nu}^2 - \frac{1}{4} F^2 g_{\mu\nu}) \\ &\quad + \frac{2}{3} (A_\mu A_\nu - \frac{1}{2} A^2 g_{\mu\nu}) - \alpha (2 \nabla^\rho \nabla^\sigma + R^{\rho\sigma}) C_{\mu\rho\sigma\nu}. \end{aligned} \quad (6)$$

It was shown in [24] that the theory admits a supersymmetric AdS vacuum with the cosmological constant  $\Lambda$ . The linear spectrum on the AdS background was analyzed. The gravity modes are identical to those in Einstein-Weyl gravity studied in [28]. (See also [29, 30, 31, 32].) There is a ghost massive spin-2 mode in addition to the massless graviton. The mass is determined by the quantity  $\alpha\Lambda$ , the product of the cosmological constant and the coupling of the Weyl-squared term. There is a critical phenomenon when

$$\alpha\Lambda = \frac{3}{2}, \quad (7)$$

for which the massive spin-2 mode disappears and is replaced by the log mode [28]. It was shown in [19] that there is no Lifshitz solution in critical Einstein-Weyl pure gravity. As we shall demonstrate presently, critical Einstein-Weyl supergravity can admit a  $z = 4$  supersymmetric Lifshitz solution owing to the existence of the massive vector field  $A_{(1)}$ .

It is worth remarking that owing to the fact that the theory is off-shell, the supersymmetric transformation rule (4) is independent of the parameter  $\alpha$ , even though the equations of motion are clearly modified by the  $\alpha$  term. This in particular means that the  $\alpha$  terms alone, which form conformal gravity, are supersymmetric. The full bosonic Lagrangian of off-shell conformal supergravity is given by

$$e^{-1} \mathcal{L} = \frac{1}{2} \alpha C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{1}{3} \alpha F_{(2)}^2, \quad (8)$$

with the supersymmetric transformation given by (4). At the first sight, it appears that the conformal symmetry is broken by the fermions. However, in this case the complex scalar  $S + iP$  couples only to the fermion and there are no bosonic equations of motion to constrain its value. Thus, they can be used to restore the conformal symmetry in the fermionic sector. Also note that in the bosonic sector, the vector field becomes massless and hence the Lagrangian (8) is gauge invariant. However, it is easy to see from (4) that the gauge invariance is broken by the fermionic sector. Thus a supersymmetric solution requires a specific gauge choice.

### 3 Supersymmetric Lifshitz and AdS vacua

In [24], it was shown that off-shell Einstein-Weyl supergravity admits an AdS vacuum, and its linear spectrum was also discussed. In a recent paper [19], it was shown that Einstein-Weyl pure gravity admits Lifshitz solutions. In this section, we construct supersymmetric Lifshitz vacua. We consider the following ansatz

$$ds^2 = \ell^2 \left( -r^{2z} dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2} \right), \quad A_{(1)} = qr^z dt. \quad (9)$$

It is clear the vector field preserves the Lifshitz symmetry. The equations of motion for the massive vector field  $A_{(1)}$  imply that

$$q(1 + \frac{2\alpha z}{\ell^2}) = 0. \quad (10)$$

Thus there is a bifurcation of solutions

$$q = 0, \quad \text{or} \quad \alpha = -\frac{\ell^2}{2z}. \quad (11)$$

For  $q = 0$ , we recover the Lifshitz solutions constructed in [19], given by

$$\alpha = \frac{3\ell^2}{2z(z-4)}, \quad \Lambda = -\frac{3+2z+z^2}{2\ell^2}. \quad (12)$$

Alternatively, for non-vanishing  $q$ , we have

$$\alpha = -\frac{\ell^2}{2z}, \quad \Lambda = -\frac{(2+z)^2}{3\ell^2}, \quad q = z-1, \quad S = \frac{z+2}{\ell}. \quad (13)$$

Note that when  $z = 1$ , the solutions of both cases reduce to the  $\text{AdS}_4$ . However, the  $\text{AdS}_4$  vacuum is a solution independent of the value of  $\alpha$ .

We now examine the supersymmetry of the Lifshitz solution (13) by solving for the Killing spinors of  $\delta\psi_\mu = 0$  in this background. First let us choose a convenient choice for the vielbein

$$e^{\hat{0}} = \ell r^z dt, \quad e^{\hat{x}} = \ell r dx, \quad e^{\hat{y}} = \ell r dy, \quad e^{\hat{r}} = \ell \frac{dr}{r}, \quad (14)$$

where we use the hatted indices to denote those of tangent space. The non-vanishing components of the corresponding spin connection are then given by

$$\omega^{\hat{0}}_{\hat{r}} = \ell^{-1} z e^{\hat{0}}, \quad \omega^{\hat{x}}_{\hat{r}} = \ell^{-1} e^{\hat{x}}, \quad \omega^{\hat{y}}_{\hat{r}} = \ell^{-1} e^{\hat{y}}. \quad (15)$$

The Killing spinors satisfy

$$\mu = 0 : \quad \left( -\frac{1}{2} z \Gamma_r + \frac{1}{3} (z-1) \Gamma^{\hat{0}} \Gamma_5 - \frac{1}{6} (z+2) \right) \epsilon = 0,$$

$$\begin{aligned}
\mu = x, y : \quad & \left( -\frac{1}{2}\Gamma_r - \frac{1}{6}(z-1)\Gamma^{\hat{0}}\Gamma_5 - \frac{1}{6}(z+2) \right) \epsilon = 0, \\
\mu = r : \quad & \left( -\partial_r + \frac{z-1}{6r}\Gamma_{\hat{r}}\Gamma_5\Gamma^{\hat{0}} - \frac{z+2}{6r}\Gamma_{\hat{r}} \right) \epsilon = 0.
\end{aligned} \tag{16}$$

A solution exists and it is given by

$$\epsilon = \sqrt{r} \epsilon_0, \quad \Gamma_{\hat{r}}\epsilon_0 + \epsilon_0 = 0, \quad \Gamma^{\hat{0}}\Gamma_5\epsilon_0 + \epsilon_0 = 0, \tag{17}$$

Thus we see that there are two independent and commuting projections on the constant spinors  $\epsilon_0$ , yielding the conclusion that the Lifshitz solutions preserve  $\frac{1}{4}$  of the supersymmetry. Note that for  $z = 1$ , there is a supersymmetric enhancement since the solutions becomes  $\text{AdS}_4$ , which also admits additional Killing spinors that depend on  $t, x, y$  coordinates [25].

In the limit of  $\alpha \rightarrow \infty$ , we have  $z = 0$ . The solution can be viewed as that of conformal supergravity. The solution is somewhat trivial since the vector becomes massless and pure gauge. The metric is a direct product of time and a hyperbolic 3-space.

The analogous demonstration can be used to show that the Lifshitz solution (12) of Einstein-Weyl pure gravity with the vanishing massive vector has no Killing spinor. This is because the metric and hence the spin connection is of the same form as that with  $A_{(1)}$  turned on. It follows that the contribution in the Killing spinor equations due to the spin connection is linear to  $z$ , and hence it cannot balance the contribution from the cosmological term which is square root of a quadratic function of  $z$ .

Thus we see that there is no smooth deformation that connects to the supersymmetric (13) and non-supersymmetric (12) Lifshitz solutions in Einstein-Weyl supergravity. This is related the fact that the coupling for the massive vector is associated with the coupling of the Weyl-squared term by the supersymmetry. If we relax this condition and consider the following Lagrangian:

$$e^{-1}\mathcal{L} = R - \frac{2}{3}(A_{(1)}^2 + S^2 + P^2) + 4S\sqrt{-\Lambda/3} + \frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{3}\beta F_{(2)}^2, \tag{18}$$

We then have a continuous family of solutions

$$\alpha = -\frac{\ell^2(3-3z+q^2)}{2z(4-5z+z^2)}, \quad \beta = -\frac{\ell^2}{2z}, \quad \Lambda = \frac{q^2-3(3+2z+z^2)}{6\ell^2}. \tag{19}$$

This general solutions contain the ones of Einstein-Weyl pure gravity (with  $q = 0$ ) [19] and cosmological Einstein gravity with a massive vector (with  $\alpha = 0$ ) [6]. Of course, the price for having such a continuous deformation is that the theory (18) with  $\beta \neq \alpha$  is not supersymmetric.

To conclude this section, we remark that the allowed Lifshitz exponents for both supersymmetric (13) and non-supersymmetric (12) solutions are determined by the quantity  $\alpha\Lambda$ .

It was known that in cosmological Einstein gravity coupled to a massive vector, the Lifshitz exponents are determined by the mass parameter. In Einstein-Weyl gravity, the mass of the massive spin-2 mode is determined by  $\alpha\Lambda$ , and it is therefore not surprising that it is also responsible for determining the Lifshitz exponents. In general there are two allowed values of  $z$  for a given  $\alpha\Lambda$ . For the non-supersymmetric solution (12), the two values coalesce when  $\alpha\Lambda = 3/2$ , the critical point. The resulting coalesced value is  $z = 1$  and hence there is no Lifshitz solution in critical Einstein-Weyl pure gravity [19]. The situation is quite different for supersymmetric Lifshitz solutions in Einstein-Weyl supergravity. The allowed values for  $z$  are given by

$$\alpha\Lambda = \frac{(z+2)^2}{6z} \quad \longrightarrow \quad z = -2 + 3\alpha\Lambda \pm \sqrt{3\alpha\Lambda(3\alpha\Lambda - 4)}. \quad (20)$$

For  $z$  to be real, we must have  $\alpha\Lambda \leq 0$  or  $\alpha\Lambda \geq 4$ . At the critical point  $\alpha\Lambda = 3/2$ , the allowed values for  $z$  are 1 and 4. Thus in addition to the  $\text{AdS}_4$  vacuum, critical supergravity allows  $z = 4$  Lifshitz solution as well, in which the massive vector  $A_{(1)}$  is turned on. The two values of  $z$  coalesce when  $\alpha\Lambda = 4/3$  for which  $z = 2$ .

It should be remarked that our solutions demonstrate explicitly that the auxiliary fields in off-shell supergravities can play a non-trivial role in constructing supersymmetric solutions once they develop dynamics after higher-order invariants are involved.

## 4 Supersymmetric asymptotic Lifshitz and AdS solutions

It was shown in [19] that Einstein-Weyl pure gravity not only admits Lifshitz vacuum solutions (12), but also black holes that are asymptotic to those Lifshitz vacua. However, owing to the complexity of the non-linear differential equations, no known examples of exact asymptotic Lifshitz solutions were constructed except in the conformal gravity limit. In this section, we consider supersymmetric asymptotic Lifshitz and also AdS solutions with the plane isometry. The existence of supersymmetry reduces the equations of motion drastically, and enables us to obtain exact solutions in some special cases. For large number of solutions, we still have to use the numerical approach even if the equations are simplified significantly.

Let us start with the more general membrane-like ansatz

$$ds^2 = \frac{dr^2}{f} - a dt^2 + r^2(dx^2 + dy^2), \quad A_{(1)} = c dt, \quad (21)$$

where the functions  $a, f$  and  $c$  depend on the coordinate  $r$  only. This ansatz contains the Lifshitz solutions (9) after some trivial scaling of the coordinates  $(x, y)$ . The vielbein and



the corresponding spin connection are chosen to be

$$\begin{aligned} e^{\hat{0}} &= \sqrt{a} dt, & e^{\hat{x}} &= r dx, & e^{\hat{y}} &= r dy, & e^{\hat{r}} &= \frac{dr}{\sqrt{f}}. \\ \omega^{\hat{0}}_{\hat{r}} &= \frac{a' \sqrt{f}}{2a}, & \omega^{\hat{x}}_{\hat{r}} &= \frac{\sqrt{f}}{r} e^{\hat{x}}, & \omega^{\hat{y}}_{\hat{r}} &= \frac{\sqrt{f}}{r} e^{\hat{y}}, \end{aligned} \quad (22)$$

Following the analogous analysis of the Killing spinor equation in the previous section, we find that the existence of a Killing spinor implies that

$$\frac{\sqrt{f} a'}{4a} - \frac{c}{3\sqrt{a}} - \frac{1}{6}S = 0, \quad \frac{\sqrt{f}}{2r} + \frac{c}{6\sqrt{a}} - \frac{1}{6}S = 0, \quad (23)$$

where  $S$  is given by (5). Thus we find that

$$c = \frac{3\sqrt{a}(\sqrt{-\Lambda/3}r - \sqrt{f})}{r}, \quad \frac{a'}{a} = \frac{2\sqrt{-3\Lambda}}{\sqrt{f}} - \frac{4}{r}. \quad (24)$$

Substituting these into the bosonic equations of motion, we find that equations are reduced to the following second-order non-linear differential equation for  $f$ :

$$\alpha \left( 3\sqrt{f}(rf'' - 5f' + 4\Lambda r) + 5\sqrt{-3\Lambda}(rf' + 2f) \right) - 2r(\sqrt{-3\Lambda}r - 3\sqrt{f}) = 0. \quad (25)$$

It is clear that if we turn off the vector field, namely by setting  $c = 0$ , we have only the AdS vacuum solution, given by

$$f = -\frac{1}{3}\Lambda r^2, \quad a = r^2. \quad (26)$$

It is also straightforward to verify that the Lifshitz solutions obtained in the previous section satisfy these equations. An important question to ask is whether an extremal black hole solution can emerge in this system. The answer is no for non-vanishing  $\Lambda$  and finite  $\alpha$ . This is because if  $a$  has a zero at  $r = r_0 > 0$ , a necessary requirement for a black hole, it follows from the second equation of (24) that the function  $f$  must have a double root at  $r_0$ . However, it follows from (25) that such a double root for  $f$  is not possible for non-vanishing  $\Lambda$ . In what follows we shall present some exact solutions in special cases and numerical solutions for general parameters.

#### 4.1 Exact solutions in Einstein-Weyl supergravity

If one is able to solve for  $f$  in (25), one can obtain  $a$  and  $c$  straightforwardly from (24). We have not obtained the general solution for (25). However, for some special parameters, we are able to obtain exact solutions.

Let us set  $\Lambda = 0$ , in which case, we find that (25) can be solved completely, giving the following local solution.

$$\begin{aligned} ds^2 &= 4\alpha \left( \frac{dr^2}{r^2 - r_0^2 + c r^6} - r^{-4} dt^2 + r^2 (dx^2 + dy^2) \right), \\ A_{(1)} &= -\frac{3}{r^3} \sqrt{r^2 - r_0^2 + c r^6} dt. \end{aligned} \quad (27)$$

In presenting the solution, we have scaled the coordinates  $t, x$  and  $y$  appropriately. The Riemann-square of the metric is given by

$$\alpha^2 \text{Riem}^2 = \frac{27}{4} + \frac{3}{4} c r^2 (9c r^6 + 10r^2 - 6r_0^2) - \frac{3r_0^2(22r^2 - 15r_0^2)}{4r^4}. \quad (28)$$

Thus we see that for vanishing  $c$  and  $r_0$ , the solution is the Lifshitz with  $z = -2$  and the metric is regular everywhere. For  $c = 0$  but  $r_0 \neq 0$ , the solution has a curvature singularity at  $r = 0$ . For  $r_0 = 0$  but  $c \neq 0$ , the singularity is located at  $\infty$ . For both non-vanishing parameters, the solution has a curvature singularity at both  $r = 0$  and  $\infty$ . The singularity at  $r = \infty$  for non-vanishing  $c$  is a null singularity, since it takes an infinity physical time for the light geodesic to reach the singularity:

$$t \sim \sqrt{c} \int^\infty \frac{dr}{r} \rightarrow \infty. \quad (29)$$

First let us consider the case with  $c = 0$ . The solution describes a smooth wormhole that connects two symmetric Lifshitz worlds with  $z = -2$ . The curvature singularity at  $r = 0$  is geodesically disconnected from the wormhole. To see this more clearly, it is instructive to write the metric in proper radial coordinate, namely

$$ds^2 = 4\alpha \left( d\rho^2 - (\cosh \rho)^{-4} dt^2 + (\cosh \rho)^2 (dx^2 + dy^2) \right), \quad A_{(1)} = \frac{3 \sinh \rho}{(\cosh \rho)^3} dt. \quad (30)$$

The solution requires that  $\alpha > 0$ , and this implies that the vector field is a ghost. If on the other hand  $\alpha < 0$ , we can perform appropriate analytic continuation  $(x, y) \rightarrow i(x, y)$  and treat the coordinate  $\rho$  as the time variable. We obtain the BPS cosmological solution

$$ds^2 = 4(-\alpha) \left( -dt^2 + (\cosh t)^{-4} dz^2 + (\cosh t)^2 (dx^2 + dy^2) \right), \quad A_{(1)} = \frac{3 \sinh t}{(\cosh t)^3} dz. \quad (31)$$

Now let us consider the solution with  $r_0 = 0$ , for which it is advantageous to use  $\rho = 1/r$  as the radial coordinate. We have

$$\begin{aligned} ds^2 &= 4\alpha \left( \frac{d\rho^2}{\rho^2 W} - \rho^4 dt^2 + \frac{1}{\rho^2} (dx^2 + dy^2) \right), \quad A_{(1)} = -3\rho^2 \sqrt{W} dt, \\ W &= 1 - \frac{a}{\rho^4}. \end{aligned} \quad (32)$$

The solution are asymptotic to the  $z = -2$  Lifshitz solution in the UV region. For  $a > 0$ , the solution is a wormhole that connects to the two asymptotic regions. for  $a < 0$ , the solution can be viewed as Lifshitz black hole, with a null singularity at  $\rho = 0$ . Since the horizon and the singularity coincides, the temperature is zero, consistent with supersymmetry. Furthermore, the vector field  $A_{(1)}$  is finite and non-vanishing at the horizon  $\rho = 0$ .

## 4.2 Exact solutions in conformal supergravity

For constant  $S$  given in (5), the equations for BPS solutions are given by (25) with  $\alpha$  sent to infinity. We find an exact solution for  $\alpha \rightarrow \infty$ , given by

$$\begin{aligned} ds^2 &= -\frac{(r-r_0)^6}{r^4} dt^2 + \frac{dr^2}{(r-r_0)^2} + r^2(dx^2 + dy^2), \\ A_{(1)} &= \frac{3r_0(r-r_0)^3}{r^3} dt, \quad S = 3. \end{aligned} \quad (33)$$

In conformal supergravity, the vector is massless and this solution describes a charged extremal black brane in conformal supergravity, with the electric charge per unit area given by

$$Q = \alpha \int F_{(2)} = 9\alpha r_0^2. \quad (34)$$

Note that the field  $S$  does not appear in the bosonic action in conformal gravity, but only in the off-shell supersymmetric transformation rules. This solution describes an extremal BPS black brane, with the coordinate  $r$  running from the  $\text{AdS}_2 \times T^2$  at the horizon  $r = r_0$  to the  $r \rightarrow \infty$  asymptotic  $\text{AdS}_4$ .

As discussed previously, in conformal supergravity, the scalar  $S$  is unrestricted, not even has to be a constant. It follows that in (23)  $S$  is a function of  $r$  rather than a parameter given in (5). For asymptotic AdS solution, we can take  $a = f$  without loss of generality in conformal gravity. This leads to the following BPS solution

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2(dx^2 + dy^2), \quad f = r^2 + c_1 r + c_0, \\ A_{(1)} &= -\frac{c_0}{r} dt, \quad S = \frac{4c_0 + 5c_1 r + 6r^2}{2r\sqrt{f}}. \end{aligned} \quad (35)$$

The Riemann-square of the metric is given by

$$\text{Riem}^2 = 24 + \frac{24c_1}{r} + \frac{8c_1 r_0}{r^3} + \frac{4r_0^2}{r^4} + \frac{8(c_1^2 + r_0)}{r^2}. \quad (36)$$

Thus the curvature singularity is located at  $r = 0$ . This solution describes a black hole with non-vanishing temperature when the function  $f$  has a single root. This clearly contradicts with the fact that non-vanishing temperature breaks supersymmetry. It turns out that

when  $f$  has a single root, the function  $S$  is divergent on the horizon, and hence the Killing spinors are not well defined there. Thus although the solution has locally-defined Killing spinor outside the horizon, the solution is not supersymmetric. When  $f$  has a double root, corresponding to  $f = (r - r_0)^2$ , we have

$$\begin{aligned} ds^2 &= -(r - r_0)^2 dt^2 + \frac{dr^2}{(r - r_0)^2} + r^2(dx^2 + dy^2), \\ A_{(1)} &= -\frac{r_0^2}{r} dt, \quad S = 3 - \frac{2r_0}{r}. \end{aligned} \quad (37)$$

Note that in this case, the function  $S$  is finite and non-vanishing on and outside the horizon. Thus we have obtained another extremal BPS black brane in the off-shell conformal supergravity. The solution runs from the  $\text{AdS}_2 \times T^2$  at the horizon  $r = r_0$  to the  $\text{AdS}_4$  at the asymptotic  $r \rightarrow \infty$ . In fact it is easy to demonstrate that this solution is conformally equivalent to the extremal AdS black hole (33). To see this, we can multiply the metric in (33) by a conformal factor  $\rho^2/r^2$  with

$$\frac{(r - r_0)^3}{r^3} = \frac{\rho - \rho_0}{\rho}, \quad (38)$$

and  $\rho_0 = 3r_0$ . The resulting metric, if expressed in terms of  $\rho$  variable, is identical to that in (37). The conformal factor is finite and non-vanishing outside and on the horizon and hence the two metrics have the same topology. (Examples of conformally locally equivalent black holes with different global structures in conformal gravity can be found in [19].)

We also obtain asymptotic Lifshitz solution, given by

$$\begin{aligned} ds^2 &= -h dt^2 + \frac{dr^2}{r^2 h} + r^2(dx^2 + dy^2), \quad h = c_0 r^2 + c_1 + \frac{c_2}{r^2}, \\ A_{(1)} &= \frac{2c_2}{r^2}, \quad S = \frac{3c_0 r^3 + 2c_1 r^2 + c_2}{r^2 \sqrt{h}}. \end{aligned} \quad (39)$$

The BPS condition requires that  $S$  being finite and hence  $h$  cannot have a single root to give rise to a black hole with temperature. When  $S$  has a double zero at  $r = r_0$ , neither  $c_0$  nor  $c_2$  vanishes, and consequently the solution has power-law singularities at both  $r = 0$  and  $r = \infty$ . For  $c_0 = 0$ , with  $c_1$  and  $c_2$  both positive, the solution is asymptotic to the Lifshitz solution with  $z = 0$ , but it suffers from having a naked power-law curvature singularity at  $r = 0$ . If instead  $c_2 = 0$ , the singularity occurs at  $r \rightarrow \infty$ .

As mentioned in the introduction, the gauge invariance of the bosonic action of conformal supergravity is broken by the fermionic sector. Thus although adding a pure gauge to  $A_{(1)}$  does not affect the bosonic equations of motion for any of the solutions discussed in this subsection, it will break the supersymmetry.

### 4.3 Numerical solutions

Although we have obtained some exact solutions for (25), it is not clear to us how to solve it in general. In this subsection, we use numerical analysis to obtain asymptotic Lifshitz and AdS solutions for generic  $\alpha\Lambda$ . As we have explained earlier, there can be no extremal black hole solutions for non-vanishing  $\Lambda$ . The only smooth solutions are wormholes that connect to two AdS or Lifshitz vacua. Analogous to (35) with  $c = 0$ , the solutions are characterized by the following behavior of the metric functions  $a$  and  $f$ . Running from  $r \rightarrow \infty$ , the function  $f$  runs from  $\sim r^2$  to encounter a single zero at the middle  $r = r_0$ , throughout which  $a$  runs from  $r^{2z}$  to a constant without being singular or vanishing.

To demonstrate that such a solution indeed exists for generic  $\alpha\Lambda$ , we first perform the Taylor expansion in the middle where the wormhole throat resides. Then we perform the Taylor expansion at the asymptotic infinity. We treat the middle expansion as the boundary condition and use the numerical analysis to show that the asymptotic infinity can indeed be reached.

#### 4.3.1 Middle expansion

Let us assume that the function  $f$  has a single largest root at  $r = r_0$  between  $r_0$  to  $\infty$ , we can perform Taylor expansion around  $r = r_0$ , given by

$$f = f_1(r - r_0) + f_2(r - r_0)^{3/2} + f_3(r - r_0)^2 + f_4(r - r_0)^{5/2} + \dots \quad (40)$$

We find that the low-lying coefficients are

$$\begin{aligned} f_2 &= \frac{4\lambda(2r_0 - 5\alpha f_1)}{3\alpha\sqrt{f_1}}, & f_3 &= -\frac{3f_2\sqrt{f_1}}{8\lambda r_0} + \frac{2\lambda^2(34\alpha^2 f_1^2 - 5\alpha f_1 r_0 - 2r_0^2)}{3\alpha^2 f_1^2}, \\ f_4 &= -\frac{\lambda}{45\alpha^3 f_1^{7/2} r_0} \left( 3\alpha f_1^2 (315\alpha^2 f_1^2 - 108\alpha f_1 r_0 - 4r_0^2) \right. \\ &\quad \left. + 4\lambda^2 r_0 (260\alpha^3 f_1^3 + 93\alpha^2 f_1^2 r_0 - 20r_0^3) \right), \end{aligned} \quad (41)$$

where we have let  $\Lambda = -3\lambda^2$ . The corresponding  $a$  is then given by

$$\begin{aligned} \frac{a}{a_0} &= 1 + a_1\sqrt{r - r_0} + a_2(r - r_0) + a_3(r - r_0)^{3/2} + \dots, \\ a_1 &= \frac{12\lambda}{\sqrt{f_1}}, & a_2 &= -\frac{4(\alpha f_1^2 - 23\alpha f_1 \lambda^2 r_0 + 2\lambda^2 r_0^2)}{\alpha f_1^2 r_0}. \end{aligned} \quad (42)$$

The function  $c$  is given by

$$c = 3\lambda r_0 + \frac{3(f_1 - 6\lambda^2 r_0)}{\sqrt{f_1}} \sqrt{r - r_0} + \dots \quad (43)$$

Thus we see that  $r = r_0$  is not a horizon, but a wormhole throat. Note that for the solution to have a double zero, we must have  $\lambda = 0$ , *i.e.* the cosmological constant  $\Lambda$  vanishes, for which exact solutions were obtained in section 4.1. To investigate the wormhole throat in some detail, we consider the co-moving coordinate  $\rho$ , defined by  $d\rho = dr/\sqrt{f}$ , namely

$$r = r_0 + \frac{1}{4}f_1\rho^2 + \frac{2r_0 - 5\alpha f_1}{12\alpha}\rho^3 + \dots, \quad (44)$$

in the middle region of  $r = r_0$ , where we have set  $\lambda = 1$ . Thus we have  $\rho \rightarrow 0$  when  $r$  approaches  $r_0$ . For the above expansion, we see that  $r$  has the minimum  $r_0$  at  $\rho = 0$  provided that  $f_1 > 0$ . This implies that the scale factor  $r^2$  of the  $T^2$  metric  $dx^2 + dy^2$  has a local minimum. The metric function associated with  $g_{tt}$  in terms of  $\rho$  in the middle is given by

$$\frac{a}{a_0} = 1 + 6\rho + \left(18 - \frac{f_1}{r_0}\right)\rho^2 + \dots. \quad (45)$$

Thus we see that  $a$  at  $\rho = 0$  is not the minimum. However, as we discussed in the preamble in section 4,  $a$  cannot be zero, and it follows that  $a$  must have the minimum  $a_{\min} > 0$  for some different  $\rho$ . We shall use numerical analysis to demonstrate that these middle expansions can be smoothly extended to the asymptotic infinities. Thus we establish that  $r = r_0$  is not only a local but also the global minimum, and hence our solutions in general describe asymmetric wormholes where the minima for  $r^2$  and  $a^2$  are located at different points of the radial coordinate. If  $\lambda = 0$ , *i.e.* the cosmological constant vanishes, the second term in (45) vanishes and the wormhole, discussed in section 4.1, is symmetric.

The half integer powers in the near horizon expansion cannot be removed for non-vanishing  $\Lambda$ . If we choose  $\alpha$  appropriately such that  $f_2 = 0$ , it is easy to verify that the coefficient  $f_4$  is non-vanishing. Such structures of the middle expansions suggest that the function  $f$  is intrinsically a complicated function and cannot be expressed in terms of a rational polynomial.

#### 4.3.2 Asymptotic expansion

For any given  $\alpha\Lambda$ , there is always an AdS vacuum solution. In addition, there can be two BPS Lifshitz solutions. In this subsection, we study the asymptotic expansions around the AdS and Lifshitz vacua. For simplicity, we shall set  $\Lambda = -3$ . Asymptotically, we have  $f = r^2/\ell^2$ . Substituting this into (25) we find

$$\ell = 1, \quad \text{or} \quad \ell = \frac{1}{3}(z + 2). \quad (46)$$

The first case gives rise to the  $\text{AdS}_4$  whilst the latter case gives rise to the Lifshitz vacuum with exponent  $z$ .

**Asymptotic AdS expansion:** The AdS vacuum solution exists for all values of the  $\alpha$  parameter. We find that the large  $r$  expansions around the asymptotic AdS for various functions are given by

$$\begin{aligned} a &= r^2 + c_+ r^{\frac{1}{2}(1+n)} + c_- r^{\frac{1}{2}(1-n)} + \dots, \\ f &= r^2 + \frac{1}{6}(3-n)c_+ r^{\frac{1}{2}(1+n)} + \frac{1}{6}(3+n)c_- r^{\frac{1}{2}(1-n)} + \dots, \\ c &= -\frac{1}{4}(3-n)c_+ r^{-\frac{1}{2}(1-n)} - \frac{1}{4}(3+n)c_- r^{-\frac{1}{2}(1+n)} + \dots, \end{aligned} \quad (47)$$

where  $n = \sqrt{(\alpha-4)/\alpha}$ . Note that when  $0 < \alpha < 4$ , the power  $n$  is complex implying some oscillatory behavior in the next leading order. The fact that the sub-leading term in the asymptotic region expansion involve generic non-integer powers suggests that a exact solution for generic  $\alpha$  is unlikely. At the critical point  $\alpha = -1/2$ , there is a more elegant AdS expansion, given by

$$\begin{aligned} a &= r^2 - \frac{2M}{r} - \frac{2M^2}{9r^4} - \frac{469M^3}{729r^7} + \dots, \\ f &= r^2 - \frac{2M}{r} - \frac{13M^2}{9r^4} - \frac{469M^3}{243r^7} \dots, \\ c &= \frac{3M}{r^2} + \frac{2M^2}{3r^5} + \frac{86M^3}{81r^8} + \dots, \end{aligned} \quad (48)$$

Thus we see that there is no supersymmetric logarithmic solution at the critical points. In contrast, non-supersymmetric Logarithmic black holes were known to exist in critical gravity [19]. Our result suggests that the supersymmetric perturbation of the AdS<sub>4</sub> vacuum is ghost free.

**Asymptotic Lifshitz expansion:** For Lifshitz solutions, the Lifshitz exponents  $z$  depends on  $\alpha$ . Since we have set  $\Lambda = -3$ , it follows that we have

$$\alpha = -\frac{(z+2)^2}{18z}. \quad (49)$$

The expansion up to the sub-leading terms are given by

$$\begin{aligned} a &= \frac{1}{9}(z+2)^2 \left[ r^{2z} + c_+ r^{-\frac{2}{3} + \frac{7}{6}z + \frac{1}{6}m} + c_- r^{-\frac{2}{3} + \frac{7}{6}z - \frac{1}{6}m} + \dots \right], \\ f &= \frac{9}{(z+2)^2} \left[ r^2 - \frac{1}{6(z+2)} \left( (m-4-5z)c_+ r^{\frac{4}{3} + \frac{m}{6} - \frac{5z}{6}} \right. \right. \\ &\quad \left. \left. - (m+4+5z)c_- r^{\frac{4}{3} - \frac{m}{6} - \frac{5z}{6}} \right) + \dots \right], \\ c &= (z-1)r^z + \frac{1}{196(z+2)} \left( (-520 + 49m + 2m^2 - 83z)c_+ r^{-\frac{2}{3} + \frac{m}{6} + \frac{z}{6}} \right. \\ &\quad \left. + (-520 - 49m + 2m^2 - 83z)c_- r^{-\frac{2}{3} - \frac{m}{6} + \frac{z}{6}} \right) + \dots, \end{aligned} \quad (50)$$

where  $m = \sqrt{64 - 32z + 49z^2}$ . It is worth noting that  $m$  is real for all real exponents  $z$ ; however, there are very few examples of  $m$  being integers for integer exponents. This suggests that exact solutions for generic  $z$  is highly unlikely.

At the critical point, there is a Lifshitz solution with  $z = 4$ ; however, the sub-leading term involve irrational powers since  $m = 12\sqrt{5}$ . The best behavior occurs at  $\alpha = -4/9$ , where the two Lifshitz exponents coalesce to become  $z = 2$ . Even in this case, the expansion powers are not of integers but rational numbers:

$$\begin{aligned} a &= \frac{16}{9} \left( r^4 - \frac{M}{r^{2/3}} - \frac{13M^2}{84r^{16/3}} - \frac{185M^3}{1176r^{10}} + \cdots \right), \\ f &= \frac{9}{16} \left( r^2 - \frac{7M}{6r^{8/3}} - \frac{73M^2}{144r^{22/3}} - \frac{1147M^3}{3024r^{12}} + \cdots \right), \\ c &= r^2 + \frac{5M}{4r^{8/3}} + \frac{65M^2}{336r^{22/3}} + \frac{1325M^3}{9408r^{12}} + \cdots. \end{aligned} \quad (51)$$

### 4.3.3 Numerical results

Having obtained both the middle and asymptotic expansions, we can use numerical analysis to verify that the middle expansion of the wormhole throat can indeed smoothly extend to the asymptotic infinity. We have checked this with general parameters. As concrete examples, we consider two cases. The first is the critical supergravity, corresponding to  $\alpha = -\frac{1}{2}$ . If we let  $r_0 = 10$ , we find that smooth solutions exist for a continuous parameter range  $\delta_- \leq f_1 \leq \delta_+$ , with  $(\delta_-, \delta_+) \sim (49.98, 81.63)$ . The solution becomes singular outside this range. When  $f_1 = \delta_-$ , the solution is asymptotic to the  $z = 4$  Lifshitz vacuum. For  $\delta_- < f_1 \leq \delta_+$ , the solution becomes asymptotic AdS. This behavior is very similar to the AdS and Lifshitz black holes obtained in [19] and the smooth solutions are all asymptotic AdS except for one which is asymptotic to the Lifshitz vacuum.

Another example is  $\alpha = -4/9$ , corresponding to having a  $z = 2$  Lifshitz solution. For  $r_0 = 10$ , we again find smooth solutions in the parameter region  $\delta_- \leq f_1 \leq \delta_+$ , with  $(\delta_-, \delta_+) \sim (67.82, 83.00)$ . However, in this case, the asymptotic AdS solution occurs only when  $f_1 = \delta_+$ . In the remaining parameter space  $\delta_- \leq f_1 < \delta_+$ , the solutions are asymptotic to the  $z = 2$  Lifshitz vacuum.

Thus in general, for a given choice of  $r_0$ , we have smooth solutions in the parameter region  $\delta_- \leq f_1 \leq \delta_+$ , for some appropriate  $(\delta_-, \delta_+)$ . Depending on the value of  $\alpha$ , the solutions can be either most asymptotic AdS or most asymptotic Lifshitz. It should be pointed out that in the numerical calculation, we can distinguish whether the solution is asymptotic AdS or Lifshitz by evaluating

$$\ell^2 = \lim_{r \rightarrow \infty} \frac{r^2}{f}, \quad (52)$$

and there can only be two discrete values given in (46). We find that in general the solutions approach these two values very slowly with the logarithmic correction in  $r$ .



## 5 Scaling Noether charge and supersymmetry

It was shown in the context cosmological gravity with a massive vector that asymptotic Lifshitz solutions have an additional conserved Noether charge [33]. This Noether charge was also found in such a solution in Einstein-Weyl pure gravity [19]. In this section, we study the Noether charge in the context of Einstein-Weyl supergravity and how the supersymmetry restricts the value of this charge.

Following the procedure of [33] and also [19], we consider the following ansatz

$$ds^2 = -\tilde{a}^2 dt^2 + d\rho^2 + \tilde{b}^2(dx^2 + dy^2), \quad A_{(1)} = \tilde{c} dt. \quad (53)$$

It is straightforward to see that the Lagrangian is invariant under the global scaling symmetry

$$\tilde{a} \rightarrow \lambda^2 \tilde{a}, \quad \tilde{b} \rightarrow \lambda^{-1} \tilde{b}, \quad \tilde{c} \rightarrow \lambda \tilde{c}. \quad (54)$$

The corresponding conserved Noether charge can be obtained straightforwardly. In terms of the coordinate system (21), the Noether charge is given by

$$\begin{aligned} N = & \frac{r\sqrt{f}(2a - ra')}{\sqrt{a}} + \frac{\alpha\sqrt{f}}{6r\sqrt{a^5}} \left( 8r^3 a^2 cc' + 6a^2 f r^3 a''' - 3a^2 r^2 (2a - ra') f'' \right. \\ & + ar^2 (10af - 14fra' + 9arf') a'' + 8a^3 f - 14ra^2 a' f + 6r^3 a'^3 f \\ & \left. + 2ra^3 f' + 4a^2 r^2 a' f' - 7ar^3 a'^2 f' \right). \end{aligned} \quad (55)$$

Substituting the supersymmetry conditions (24) into the Noether charge, we find that

$$N = \sqrt{a} \left[ \alpha \left( 3\sqrt{f}(rf'' - 5f' + 4\Lambda r) + 5\sqrt{-3\Lambda}(rf' + 2f) \right) - 2r(\sqrt{-3\Lambda}r - 3\sqrt{f}) \right], \quad (56)$$

which vanishes identically because of the equation of motion (25). Thus we see that the supersymmetry constrains the Noether charge to zero. In contrast, for non-supersymmetric Lifshitz black holes obtained in [33] and [19], the Noether charge is non-vanishing and it is constant proportional to the product of the temperature and entropy of the black hole [33, 19].

## 6 Integrability condition

One important characteristics of off-shell supergravity is that the supersymmetric transformation rules form closed super algebra without requiring the equations of motion. This implies that the integrability conditions for the Killing spinor equations in off-shell supergravities have some special properties. In both on-shell or off-shell supergravities, the integrability conditions of Killing spinor equations are not immediately related to the equations

of motion, but rather they are related to the Riemann tensor of the background. However, in on-shell supergravities, a certain  $\Gamma$ -matrix projected integrability condition can be shown to vanish identically on shell. This property no longer holds in off-shell supergravity. This is because the supersymmetric transformation rules are fixed in off-shell supergravity whilst the equations of motion can be altered by the introduction of additional super invariants. Thus it is worthwhile to study the  $\Gamma$ -matrix projected integrability condition for the Killing spinor equations. We do this by consider the generic background, and then substitute the supersymmetric Lifshitz solutions to demonstrate that the integrability conditions are indeed satisfied. The purpose of this exercise is not to check whether the solutions we have obtained are indeed supersymmetric, which has already been established. Rather it is to highlight the distinguishing features of off-shell supergravities.

The Killing spinor equation is given by

$$\delta\psi_\mu = -D_\mu\epsilon + \frac{1}{6}(2A_\mu + \Gamma_{\rho\mu}A^\rho)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5P)\epsilon = 0 \quad (57)$$

By computing  $\Gamma^\nu D_{[\nu}D_{\mu]}$  on  $\epsilon$ , we have

$$\begin{aligned} 0 &= R_{\mu\nu}\Gamma^\nu\epsilon - \frac{2}{3}\Gamma^\nu(2\nabla_{[\nu}A_{\mu]} + \Gamma_{\rho[\mu}\nabla_{\nu]}A^\rho)\Gamma_5\epsilon + \frac{2}{3}\Gamma^\nu\Gamma_{[\mu}(\nabla_{\nu]}S + i\Gamma_5\nabla_{\nu]}P)\epsilon \\ &\quad - \frac{1}{9}\Gamma^\nu[(2A_{[\mu} + \Gamma_{\rho[\mu}A^\rho)\Gamma_5 - \Gamma_{[\mu}(S + i\Gamma_5P)]][(2A_{\nu]} - \Gamma_{\nu]\sigma}A^\sigma)\Gamma_5 - \Gamma_{\nu]}(S + i\Gamma_5P)]\epsilon \\ &= \left[R_{\mu\nu} + \frac{2}{9}(A_\mu A_\nu - A_\rho A^\rho g_{\mu\nu}) + \frac{1}{3}(S^2 + P^2)g_{\mu\nu}\right]\Gamma^\nu\epsilon \\ &\quad - \frac{2}{3}\left[\frac{1}{2}F_{\nu\mu} + \nabla_{(\mu}A_{\nu)} + \frac{1}{2}\nabla_\rho A^\rho g_{\mu\nu}\right]\Gamma^\nu\Gamma_5\epsilon - \frac{1}{6}F_{\nu}{}^\rho\Gamma^\nu{}_{\rho\mu}\Gamma_5\epsilon \\ &\quad + \frac{1}{3}\Gamma^\nu{}_\mu\left[\nabla_\nu - \frac{2}{3}A_\nu\Gamma_5\right](S + iP\Gamma_5)\epsilon - \left[\nabla_\mu + \frac{2}{3}A_\mu\Gamma_5\right](S + iP\Gamma_5)\epsilon. \end{aligned} \quad (58)$$

For the leading-order supergravity with  $\alpha = 0$ , the massive vector  $A_{(1)}$  vanishes, and it is then straightforward to see that the integrability condition is satisfied by the equations of motion, without further checking. However, when the  $\alpha$  term is involved,  $A_{(1)}$  becomes dynamical and the equations of motion involve the variation of the Weyl-squared terms. It is thus clear that the above expression in the right will not vanish identically on-shell. For the solution (13), we find

$$\begin{aligned} 0 &= \ell^{-1}\frac{2}{3}(z-1)(z+2)r^z\Gamma^{\hat{0}}\epsilon - \frac{2}{3}\ell^{-1}(z-1)(z+2)r^z\Gamma_5\epsilon \\ 0 &= \frac{1}{9}(5z+4)(z-1)\ell^{-1}r\Gamma^{\hat{z}}\epsilon - \frac{1}{3}z(z-1)\ell^{-1}r\Gamma^{\hat{r}\hat{0}}\Gamma_5\epsilon - \frac{2}{9}(z-1)(z+2)\ell^{-1}r\Gamma_5\Gamma^{\hat{z}}\epsilon \\ 0 &= -\frac{4}{9}(z-1)^2\ell^{-1}r^{-1}\Gamma^{\hat{r}}\epsilon + \frac{2}{3}z(z-1)\ell^{-1}r^{-1}\Gamma^{\hat{0}}\Gamma_5\epsilon - \frac{2}{9}(z-1)(z+2)\ell^{-1}r^{-1}\Gamma_5\Gamma^{\hat{0}}\Gamma^{\hat{r}}\epsilon \end{aligned} \quad (59)$$

It implies that

$$\Gamma_{\hat{r}}\epsilon_0 + \epsilon_0 = 0, \quad \Gamma^{\hat{0}}\Gamma_5\epsilon_0 + \epsilon_0 = 0, \quad (60)$$

which are precisely the condition we have obtained in section 3. Thus we see explicitly that the  $\Gamma$ -matrix projected integrability conditions do not vanish identically by the equations of motion, but they are subject to further constraints.

Even for the leading-order theory with  $\alpha = 0$ , the structures of the  $\Gamma$ -matrix projected integrability condition (58) appear rather “unruly” and at the first sight, it is hard to see how such structures can cancel the supersymmetric variation of the bosonic Lagrangian. For example, although the first line of the right hand side of the equation (58) is indeed related to the Einstein equations of motion for  $\alpha = 0 = \Lambda$ , the remaining terms appear to be rather disorganized. To investigate the supersymmetric invariance of the Lagrangian in some detail, we present the fermionic Lagrangian up to and including the quadratic order:

$$e^{-1}\mathcal{L}_F = \bar{\psi}_\mu \Gamma^{\mu\nu\rho} \psi_{\nu\rho} - 2\sqrt{-\Lambda/3} \bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu - \frac{2}{3}\alpha \left( \bar{\psi}_{\mu\nu} \Gamma^\rho D_\rho \psi_{\mu\nu} - \bar{\psi}_{\mu\lambda} \Gamma^{\mu\nu\rho} D_\rho \psi_\nu{}^\lambda + \dots \right), \quad (61)$$

where  $\psi_{\mu\nu} = 2D_{[\mu}\psi_{\nu]}$  and the ellipses inside the last bracket denote the quadratic fermion terms of the type  $\psi^2 \times \nabla(\text{bosonic fields})$ . Note that for  $\alpha = 0$ , the auxiliary fields do not appear at all in the quadratic fermionic action. The supersymmetric transformation rule for the gravitino was given in (4) and the ones for the bosonic fields are [19]

$$\begin{aligned} \delta e_\mu{}^a &= \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta S &= \frac{1}{2} \bar{\epsilon} \Gamma^{\mu\nu} \widehat{\psi}_{\mu\nu} \\ &= \frac{1}{2} \bar{\epsilon} [\Gamma^{\mu\nu} \psi_{\mu\nu} + A^\nu \Gamma_5 \psi_\nu - (S + i\Gamma_5 P) \Gamma^\nu \psi_\nu], \\ \delta P &= \frac{1}{2} i \bar{\epsilon} \Gamma^{\mu\nu} \Gamma_5 \widehat{\psi}_{\mu\nu} \\ &= \frac{1}{2} i \bar{\epsilon} \Gamma_5 [\Gamma^{\mu\nu} \psi_{\mu\nu} + A^\nu \Gamma_5 \psi_\nu - (S + i\Gamma_5 P) \Gamma^\nu \psi_\nu], \\ \delta A_\mu &= \frac{1}{8} \bar{\epsilon} (\Gamma_\mu \Gamma^{\nu\rho} - 3\Gamma^{\nu\rho} \Gamma_\mu) \Gamma_5 \widetilde{\psi}_{\mu\nu} \\ &= \bar{\epsilon} (\delta_\mu^{[\nu} \Gamma^{\rho]} - \frac{1}{4} \Gamma_\mu{}^{\nu\rho}) \Gamma_5 \psi_{\nu\rho} + \bar{\epsilon} \left( -\frac{1}{2} A_\mu \Gamma^\rho + \delta_\mu^\rho A_\nu \Gamma^\nu - \frac{1}{4} A^\sigma \Gamma^\rho{}_{\sigma\mu} \right) \psi_\rho \\ &\quad - \frac{1}{2} (S \Gamma_5 + iP) \psi_\mu, \end{aligned} \quad (62)$$

where

$$\begin{aligned} \widehat{\psi}_{\mu\nu} &= \psi_{\mu\nu} + \frac{1}{3} \Gamma_5 (2A_{[\mu} + A^\rho \Gamma_{\rho[\mu} \psi_{\nu]} + \frac{1}{3} \Gamma_{[\mu} (S + i\Gamma_5 P) \psi_{\nu]}), \\ \widetilde{\psi}_{\mu\nu} &= \psi_{\mu\nu} - \frac{1}{3} \Gamma_5 (4A_{[\mu} - A^\rho \Gamma_{\rho[\mu} \psi_{\nu]} - \frac{1}{3} \Gamma_{[\mu} (S + i\Gamma_5 P) \psi_{\nu]}). \end{aligned} \quad (63)$$

Thus we see that the supersymmetric transformation rule of the bosonic field in the on-shell supermultiplet, namely the vielbein, is standard and it does not involve a derivative of gravitino. This is a generic feature for on-shell supergravities, and it is responsible for the familiar structure of the first line in the right hand side of equation (58). On the

other hand, the supersymmetric transformation rules for the auxiliary fields  $S, P$  and  $A_\mu$  all involve the gravitino field strength, a derivative of the gravitino. This implies that the mechanism for the supersymmetric invariance of the Lagrangian in off-shell supergravities is rather different from that in on-shell supergravities. To elaborate this further, let us consider the following

$$\begin{aligned}
-\Gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho &= \frac{1}{2}\left[R^\mu{}_\rho - \frac{1}{2}\delta^\mu_\rho R + \frac{1}{9}(2A^\mu A_\rho + A_\lambda A^\lambda \delta^\mu_\rho) - \frac{1}{3}(S^2 + P^2)\delta^\mu_\rho\right]\Gamma^\rho\epsilon \\
&\quad - \frac{1}{12}F_{\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_5\epsilon - \frac{1}{3}\left[\nabla_{(\nu}A_{\rho)} - \nabla_\lambda A^\lambda g_{\nu\rho} - \frac{1}{2}F^\mu{}_\rho\right]\Gamma^\rho\Gamma_5\epsilon \\
&\quad + \frac{1}{3}\Gamma^{\mu\rho}\left[\nabla_\rho + \frac{1}{3}A_\rho\Gamma_5\right](S + iP\Gamma_5)\epsilon - \frac{1}{3}A^\mu\Gamma_5(S + iP\Gamma_5)\epsilon \\
&\quad - \frac{1}{6}\left[(\Gamma^{\mu\nu\rho}A_\rho + 4A^{[\mu}\Gamma^{\nu]})\Gamma_5 - 2\Gamma^{\mu\nu}(S + iP\Gamma_5)\right]\delta\psi_\nu. \tag{64}
\end{aligned}$$

In on-shell supergravities, the above quantities involving the manifest  $\epsilon$  terms would precisely cancel the variation of the bosonic Lagrangian whilst the  $\delta\psi_\nu$  term would cancel the variation of the Yukawa terms in supergravities.<sup>1</sup> This clearly is not true for (64), even though the first term in the right hand side of (64) indeed cancels the variation of bosonic Lagrangian with  $\alpha = 0 = \Lambda$ . It is thus worth checking explicitly how the Lagrangian is invariant under the supersymmetry.

The Lagrangian for Einstein-Weyl supergravity we have considered in this paper in fact contains three independent invariants [19]

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^{(1)} + 2\sqrt{-\Lambda/3}\mathcal{L}^{(2)} + \frac{1}{2}\alpha\mathcal{L}^{(3)}, \\
\mathcal{L}^{(1)} &= \mathcal{L}_B^{(1)} + \mathcal{L}_F^{(1)} = \sqrt{-g}\left(R - \frac{2}{3}(A_{(1)}^2 + S^2 + P^2)\right) + \sqrt{-g}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\psi_{\nu\rho}, \\
\mathcal{L}^{(2)} &= \mathcal{L}_B^{(2)} + \mathcal{L}_F^{(2)} = 2\sqrt{-g}S - \sqrt{-g}\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu, \\
\mathcal{L}^{(3)} &= \mathcal{L}_B^{(3)} + \mathcal{L}_F^{(3)} = \sqrt{-g}\left(C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{3}F_{(2)}^2\right), \\
&\quad - \frac{1}{3}\sqrt{-g}\left(\bar{\psi}_{\mu\nu}\Gamma^\rho D_\rho\psi_{\mu\nu} - \bar{\psi}_{\mu\lambda}\Gamma^{\mu\nu\rho}D_\rho\psi_\nu{}^\lambda + \dots\right). \tag{65}
\end{aligned}$$

For simplicity and the purpose of demonstrating the distinguishing feature of off-shell supergravity, we shall present only the supersymmetric invariance of the action associated with  $\mathcal{L}^{(1)}$  and  $\mathcal{L}^{(2)}$  up to and including the quadratic order in fermion. We find

$$\int d^4x \delta(\sqrt{-g}\mathcal{L}_B^{(1)})$$

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<sup>1</sup>For this reason, it was shown that a bosonic gravity system that has consistent Killing spinor equations can be promoted to pseudo-supergravity whose action, up to and including quadratic order in fermions, is invariant under the pseudo-supersymmetry [34]-[38]. If the field contents coincide with a known supermultiplet, the theory becomes indeed true supergravity. On the other hand, if the field contents do not form a supermultiplet, the pseudo-supersymmetric transformation rules will not form a known (finite-dimensional) super-Poincaré algebra, and the corresponding theory is called pseudo-supergravity [34]-[38].

$$\begin{aligned}
&= \int d^4x \sqrt{-g} \left( -2 \left[ R^{\mu\rho} - \frac{2}{3} A^\mu A^\rho - \frac{1}{2} g^{\mu\rho} \left( R - \frac{2}{3} (A^2 + S^2 + P^2) \right) \right] \bar{\epsilon} \Gamma_\rho \psi_\mu \right. \\
&\quad \left. - \frac{4}{3} \bar{\epsilon} (A^{[\mu} \Gamma^{\nu]} - \frac{1}{4} A_\rho \Gamma^{\rho\mu\nu}) \Gamma_5 \tilde{\psi}_{\mu\nu} - \frac{2}{3} \bar{\epsilon} \Gamma^{\mu\nu} (S + i\Gamma_5 P) \hat{\psi}_{\mu\nu} \right), \\
&\int d^4x \delta(\sqrt{-g} \mathcal{L}_F^{(1)}) \\
&= \int d^4x \sqrt{-g} \left( 2 \left( R^\mu{}_\rho - \frac{1}{2} \delta^\mu_\rho R \right) \bar{\epsilon} \Gamma^\rho \psi_\mu + \frac{4}{3} \bar{\epsilon} \left( A^\mu \Gamma^\nu - \frac{1}{4} A_\rho \Gamma^{\mu\nu\rho} \right) \Gamma_5 \psi_{\mu\nu} \right. \\
&\quad \left. + \frac{2}{3} \bar{\epsilon} \Gamma^{\mu\nu} (S + i\Gamma_5 P) \psi_{\mu\nu} \right), \\
&\int d^4x \delta(\sqrt{-g} \mathcal{L}_B^{(2)}) = \int d^4x \sqrt{-g} \left( 2S \bar{\epsilon} \Gamma^\mu \psi_\mu + \bar{\epsilon} \Gamma^{\mu\nu} \hat{\psi}_{\mu\nu} \right), \\
&\int d^4x \delta(\sqrt{-g} \mathcal{L}_F^{(2)}) = \int d^4x \sqrt{-g} \left( -\bar{\epsilon} \Gamma^{\mu\nu} \psi_{\mu\nu} - \bar{\epsilon} \left[ A^\mu \Gamma_5 + \Gamma^\mu (S - i\Gamma_5 P) \right] \psi_\mu \right). \quad (66)
\end{aligned}$$

In the above, we have performed appropriate integration by parts and dropped the surface terms. It is now a straightforward calculation to verify that the each action associated with  $\mathcal{L}^{(1)}$  and  $\mathcal{L}^{(2)}$  is independently invariant under the supersymmetry. It becomes clear that the cancelation of the last term in (64) is not by the Yukawa term, as would be the case in on-shell supergravity, but due to the gravitino field strength that appears in the transformation rules of the auxiliary fields  $S, P$  and  $A_{(1)}$ .

## 7 Spherically-symmetric solutions

The asymptotic AdS and Lifshitz solutions we have obtained so far are characterized by the flat level surfaces and the metric is brane like. It is natural to investigate whether there are spherically-symmetric solutions that are also supersymmetric. Spherically-symmetric AdS and Lifshitz-like black holes were shown to exist in both Einstein-Weyl and conformal gravities [19]. Supersymmetric and static asymptotic AdS solutions are common occurrence in gauged supergravities, although they typically suffer from having a naked power-law curvature singularity. In this section, we study the supersymmetry of spherically-symmetric ansatz and demonstrate that the only supersymmetric such a solution is the AdS<sub>4</sub> vacuum. There are no asymptotic AdS or Lifshitz-like solutions that are both spherically-symmetric and supersymmetric in Einstein-Weyl or conformal supergravities.

Let us consider the ansatz:

$$ds^2 = \frac{dr^2}{f} - a dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad A_{(1)} = c dt, \quad (67)$$

where the functions  $a, f$  and  $c$  depend on  $r$  only. A natural choice for the vielbein is

$$e^{\hat{0}} = \sqrt{a} dt, \quad e^{\hat{1}} = r d\theta, \quad e^{\hat{2}} = r \sin \theta d\phi, \quad e^{\hat{r}} = \frac{dr}{\sqrt{f}}. \quad (68)$$

The corresponding spin connection has the following non-vanishing components

$$\omega^{\hat{0}}_{\hat{r}} = \frac{a'\sqrt{f}}{2a}e^{\hat{0}}, \quad \omega^{\hat{1}}_{\hat{r}} = \frac{\sqrt{f}}{r}e^{\hat{1}}, \quad \omega^{\hat{2}}_{\hat{r}} = \frac{\sqrt{f}}{r}e^{\hat{2}}, \quad \omega^{\hat{2}}_{\hat{1}} = \frac{\cos\theta}{r\sin\theta}e^{\hat{2}}, \quad (69)$$

We decompose the  $\Gamma$ -matrices as follows

$$\Gamma_{\hat{0}} = i\sigma_1 \otimes \sigma_3, \quad \Gamma_{\hat{r}} = \sigma_2 \otimes \sigma_3, \quad \Gamma_{\hat{a}} = 1 \otimes \sigma_{\hat{a}}, \quad \Gamma_5 = i\Gamma^{\hat{0}\hat{1}\hat{2}\hat{r}} = \sigma_3 \otimes \sigma_3, \quad (70)$$

where  $\sigma_i$  are Pauli matrices. We now perform the reduction on  $S^2$  by introducing two component spinor  $\chi_{\pm}$  which obey the equations

$$\nabla'_a \chi_{\pm} = \pm \frac{i}{2} \sigma_{\hat{a}} \chi_{\pm}, \quad (71)$$

where  $\nabla'$  is defined with the spin connection on the unit sphere. It is related to the covariant derivative on the spinor in the sphere direction of our ansatz as

$$\nabla_a = \nabla'_a + \frac{1}{4} \Omega_{abr} \Gamma^{br} = \nabla'_a + \frac{\sqrt{f}}{4r} \Gamma_a^{\hat{r}}, \quad (72)$$

Covariance under  $SU(2)$  transformations ensures that we can take

$$\chi_- = i\sigma_3 \chi_+ \quad (73)$$

without loss of generality. Now we can expand the spinor  $\epsilon$  over basis on the sphere

$$\epsilon = \sum_{\alpha=+,-} \epsilon_{\alpha} \otimes \chi_{\alpha}. \quad (74)$$

The Killing spinor equation in the  $t$ ,  $r$  and the sphere directions implies that

$$\begin{aligned} 0 &= \sum_{\alpha} \left[ \partial_r + \frac{c}{6\sqrt{a}} \Gamma_r^{\hat{0}} \Gamma_5 \epsilon + \frac{1}{6} \Gamma_r S \right] \epsilon_{\alpha} \otimes \chi_{\alpha} \\ &= \sum_{\alpha} \left[ \partial_r \epsilon_{\alpha} \otimes \chi_{\alpha} + \frac{c}{6\sqrt{a}f} \epsilon_{\alpha} \otimes \sigma_3 \chi_{\alpha} + \frac{1}{6\sqrt{f}} S \sigma_2 \epsilon_{\alpha} \otimes \sigma_3 \chi_{\alpha} \right], \\ 0 &= \sum_{\alpha} \left[ \partial_t + \frac{a'\sqrt{f}}{4a} \Gamma_t^{\hat{r}} - \frac{c}{3} \Gamma_5 \epsilon + \frac{1}{6} \Gamma_t S \right] \epsilon_{\alpha} \otimes \chi_{\alpha} \\ &= \sum_{\alpha} \left[ \partial_t \epsilon_{\alpha} \otimes \chi_{\alpha} + \frac{a'\sqrt{f}}{4\sqrt{a}} \sigma_3 \epsilon_{\alpha} \otimes \chi_{\alpha} - \frac{c}{3} \sigma_3 \epsilon_{\alpha} \otimes \sigma_3 \chi_{\alpha} - \frac{i\sqrt{a}}{6} S \sigma_1 \epsilon_{\alpha} \otimes \sigma_3 \chi_{\alpha} \right], \\ 0 &= \sum_{\alpha} \left[ \nabla'_a + \frac{\sqrt{f}}{2r} \Gamma_a^{\hat{r}} + \frac{c}{6\sqrt{a}} \Gamma_a^{\hat{0}} \Gamma_5 \epsilon + \frac{1}{6} \Gamma_a S \right] \epsilon_{\alpha} \otimes \chi_{\alpha} \\ &= \sum_{\alpha} e_a^{\hat{a}} \left[ \frac{i}{2r} \alpha \epsilon_{\alpha} \otimes \sigma_{\hat{a}} \chi_{\alpha} + \frac{\sqrt{f}}{2r} \sigma_2 \epsilon_{\alpha} \otimes \sigma_{\hat{a}} \sigma_3 \chi_{\alpha} \right. \\ &\quad \left. - \frac{c}{6\sqrt{a}} \sigma_2 \epsilon_{\alpha} \otimes \sigma_{\hat{a}} \chi_{\alpha} + \frac{1}{6} S \epsilon_{\alpha} \otimes \sigma_{\hat{a}} \chi_{\alpha} \right], \end{aligned} \quad (75)$$

where  $S = 3\sqrt{-\Lambda/3}$ . It follows that

$$\partial_r \epsilon_+ + \frac{ic}{6\sqrt{a}f} \epsilon_- + \frac{i}{6\sqrt{f}} S \sigma_2 \epsilon_- = 0,$$

$$\begin{aligned}
& \partial_r \epsilon_- - \frac{i c}{6\sqrt{a f}} \epsilon_+ - \frac{i}{6\sqrt{f}} S \sigma_2 \epsilon_+ = 0, \\
& \partial_t \epsilon_+ + \frac{a' \sqrt{f}}{4\sqrt{a}} \sigma_3 \epsilon_+ - \frac{i c}{3} \sigma_3 \epsilon_- + \frac{\sqrt{a}}{6} S \sigma_1 \epsilon_- = 0, \\
& \partial_t \epsilon_- + \frac{a' \sqrt{f}}{4\sqrt{a}} \sigma_3 \epsilon_- + \frac{i c}{3} \sigma_3 \epsilon_+ - \frac{\sqrt{a}}{6} S \sigma_1 \epsilon_+ = 0, \\
& \frac{i}{2r} \epsilon_+ + \frac{i \sqrt{f}}{2r} \sigma_2 \epsilon_- - \frac{c}{6\sqrt{a}} \sigma_2 \epsilon_+ + \frac{1}{6} S \epsilon_+ = 0, \\
& -\frac{i}{2r} \epsilon_- - \frac{i \sqrt{f}}{2r} \sigma_2 \epsilon_+ - \frac{c}{6\sqrt{a}} \sigma_2 \epsilon_- + \frac{1}{6} S \epsilon_- = 0,
\end{aligned} \tag{76}$$

It is clear that the Killing spinors can be expressed as  $\epsilon_{\pm} = e^{iE_{\pm}t} \tilde{\epsilon}_{\pm}(r)$ . It follows that the last four equations of (76) are effectively eight linear equations for the four variables  $\tilde{\epsilon}_{\pm}^i$  with  $i = 1, 2$ . For  $c = 0$ , we find that a solution exists, giving rise to precisely the  $\text{AdS}_4$  vacuum with the cosmological constant  $\Lambda$ . When  $c$  is non-vanishing, we find that the Killing spinor equations simply has no solution before even checking whether the background satisfies the bosonic equations of motion.

It is rather intriguing that Einstein-Weyl or conformal off-shell supergravities favor the existence of AdS and Lifshitz solutions with the plane isometry rather than the spherically-symmetric solutions. In usual supergravities, the opposite is true: the number of BPS spherically-symmetric solutions is far greater than that of the supersymmetric Lifshitz solutions which typically involve non-trivial hypermultiplets.

## 8 Conclusions

In this paper, we have studied Einstein-Weyl supergravity and conformal supergravity. The supersymmetry are realized in the off-shell formalism. The field content of the theory consists of the metric, a massive vector and one complex scalar as the bosonic fields and a gravitino field as their supersymmetric partner. In the off-shell formalism, the closure of the supersymmetry transformation rules is realized without needing the input of the equations of motion. This implies that one can alter the Lagrangian and hence the equations of motion by adding new super invariants without changing the supersymmetry. It follows that in detail calculation, the supersymmetric invariance of the action is realized very differently than that in the usual on-shell supergravities. In particular, the  $\Gamma$ -matrix projected integrability condition for the Killing spinor equations does not vanish identically on shell. Ostensibly, the supersymmetric invariance of the action is unlikely to be true. The invariance is saved by the fact that the supersymmetric transformation rules for the auxiliary fields, namely the massive vector and the complex scalar, involve derivative of the gravitino. We present the

explicit demonstration of the invariance up to and including the quadratic order of fermions in the action.

The main purpose of the paper is to construct supersymmetric solutions in Einstein-Weyl and conformal supergravities. We find that in addition to the previously-known supersymmetric AdS vacuum, the theory admits a large class of BPS Lifshitz vacua supported by the massive vector. The Lifshitz exponents  $z$  are solely determined by the product of the cosmological constant and the coupling of the Weyl-squared term. We demonstrate by obtaining explicit Killing spinors that these solutions preserve  $\frac{1}{4}$  of the supersymmetry. The result is intriguing in that the massive vector field belongs to the auxiliary fields in the off-shell supermultiplet. For the lowest-order supergravity, these fields are set to zero by the equations of motion. Thus one might have expected that these fields play no role in physics. However, when higher-derivative super invariants are added into the Lagrangian, the auxiliary fields can pick up kinetic terms and become dynamical. Our solutions are the first examples that not only these auxiliary fields in off-shell supergravity can be used to construct non-trivial solutions, they also play an essential role for the solutions to be supersymmetric. Since the auxiliary fields are not related to the central charge of the super algebra, the supersymmetry of our solutions is not realized by the standard BPS condition that balances the energy and the central charge. Instead, it is likely the case that the massive spin-2 hair balances the massive vector hair such that the Noether charge associated with the scaling symmetry of the plane-symmetric ansatz vanishes. Thus our solutions demonstrate a new mechanism of achieving supersymmetry of a bosonic background in higher-derivative off-shell supergravities.

We then obtain the equations of backgrounds that are supersymmetric and asymptotic to the AdS and Lifshitz solutions. Although the theory involve fourth-order derivatives, the equations of motion constrained by the supersymmetric are significantly simpler and reduced to one second-order non-linear differential equation. For some special choice of parameters, we have obtained many examples of exact solutions, including asymptotic Lifshitz solutions with  $z = -2$  and extremal AdS black holes. We then use numerical analysis to study the structure of general solutions. We find that for non-vanishing cosmological constant, the general solutions are wormholes that are asymptotic to Lifshitz vacua. For a given size of the wormhole throat, we find that there exists a parameter region in which the solutions are smooth. Depending on the value of the product of the cosmological constant and the coupling of the Weyl-squared term, the smooth wormholes are either mostly asymptotic AdS with the asymptotic Lifshitz solution lying at one boundary of the parameter space,



or the vice versa. We also study the supersymmetric condition of spherically-symmetric solutions and we find that the only solution is the AdS vacuum.

Higher-derivative gravity in general suffers from having ghost modes in its spectrum. Einstein-Weyl supergravity is not different [19]. However, such modes are not disastrous in condensed matter physics. Moreover, the supersymmetry that our solutions possess might help to stabilize the solutions. For example, we have demonstrated that the ghost log mode in critical supergravity does not contribute to the supersymmetric asymptotic AdS solution. The large number of supersymmetric solutions that are asymptotic to the AdS and Lifshitz vacua we have constructed in this paper demonstrate that higher-derivative off-shell supergravities can have important applications in the area of the AdS/CFT correspondence.

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